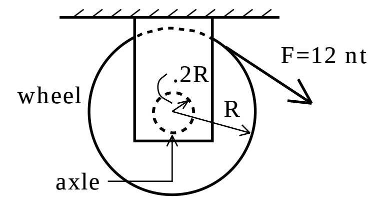
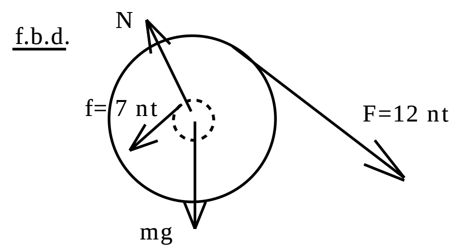
CHAPTER 9 -- ROTATIONAL MOTION II



9.1) A sketch for this set-up is shown to the right, complete with f.b.d. for the forces acting on the system. Notice that although the frictional force acts everywhere on the axle, it provides nothing more than a torque on the system. For that reason, we can assume that the net frictional force acts *at one place only* (it doesn't matter how it's oriented--*x* and *y components* aren't relevant when determining torques). Also, the normal force *N* at the pin, which does not provide a torque to the system, cannot be vertical as *F* has *x* and *y components* that *N* must counter.



a.) The request for the magnitude of the *angular acceleration* should bring to mind N.S.L. Using the rotational counterpart of that law and assuming the pulley is a disk whose $I_{cm} = (1/2)mR^2$, we get:

$\underline{\sum \Gamma_{pin}}$:

$$\Gamma_f \;+\; \Gamma_F = \qquad I \qquad \alpha$$
$$f(.2R) - FR = -\,[(1/2)MR^2]\;\alpha.$$

Dividing out an *R* yields:

$$(7\text{ nt})(.2) - (12\text{ nt}) = -.5(8\text{ kg})(.6\text{ m})\alpha$$
$$\Rightarrow \quad \alpha = 4.42\text{ rad/sec}^2.$$

b.) The relationship between the rotating object's *angular acceleration* α and the *instantaneous translational acceleration a* of a point *(2/3)R* from the axis of rotation is $a = r\alpha$. Noting that *r* is numerically equal to *(2/3)R*, but that *r's* units are *meters/radian*, we get:

$$\begin{aligned} a &= r\alpha \\ &= (2R/3)\alpha \\ &= .67(.6\text{ m/rad})(4.42\text{ rad/s}^2) \\ &= 1.78\text{ m/s}^2. \end{aligned}$$

c.) The rotational counterpart to Newton's Second Law, written in terms of angular momentum, is "the *sum of the torques* equal to the *change of the angular momentum with time*," or $\Gamma_{net} = \Delta L/\Delta t$. Using this:

$\underline{\sum \Gamma_{pin}}$:

$$\Gamma_f + \Gamma_F = \Delta L \quad / \Delta t$$
$$f(.2R) - FR = (L_5 - L_0) / t.$$

As the initial *angular momentum* is zero (the wheel is initially at rest), multiplying both sides by *t* gives us:

$$[(7\text{ nt})(.2)(.6\text{ m}) - (12\text{ nt})(.6\text{ m})]t = L_5$$
$$\Rightarrow \quad L_5 = [(7\text{ nt})(.12\text{ m}) - (7.2\text{ nt}\cdot\text{m})][5\text{ sec}]$$
$$= -31.8\text{ kg}\cdot\text{m}^2/\text{s}.$$

Note: The negative sign means the rotation is clockwise.

d.) Using the rotational form to *angular momentum*, we know that $L = I\omega$. Using that relationship to determine the angular velocity at $t = 5$ *seconds*, we find:

$$L_5 = I\omega_5$$
$$\Rightarrow \quad \omega_5 = L_5/(.5MR^2)$$
$$= (-31.8\text{ kg}\cdot\text{m}^2/\text{s})/[.5(8\text{ kg})(.6\text{ m})^2]$$
$$= -22.08\text{ rad/sec}.$$

e.) Whenever you are asked to determine an *angular VELOCITY*, think conservation of energy (at least to start). Note that whereas the work done by a force F as a body displaces a distance d is $\boldsymbol{F\cdot d}$ (equal to Fd if the force and displacement are in the same direction), the work done by a torque Γ applied through an *angular displacement* $\Delta\theta$ will be $\Gamma\Delta\theta$ (remember, planar rotation is one-dimensional). Friction-produced torques will always do *negative* work. Torques that make a body's angular velocity *increase* will do *positive* work; those that make a body's angular speed *decrease* will do *negative* work. Keeping this in mind, the *conservation of energy* implies:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$0 + 0 + [-\Gamma_f \Delta\theta + \Gamma_F \Delta\theta] = (1/2)I\omega_5^{\,2} + 0$$
$$[[-(.2R)f]\Delta\theta + FR\,\Delta\theta] = .5(.5MR^2)\,\omega_5^{\,2}$$

$$[-(.2)(.6\text{ m})(7\text{ nt})(55.15\text{ rad}) + (12\text{nt})(.6\text{ m})(55.15\text{ rad}) = .25(8\text{ kg})(.6\text{ m})^2\,\omega_5^{\,2}$$

$$\Rightarrow \quad \omega_5^{\,2} = 487.16\text{ rad}^2/\text{sec}^2$$

$$\Rightarrow \quad \omega_5 = 22.07\text{ rad/sec.}$$

Note: Remember, the conservation of energy yields *magnitudes only*.

f.) Using rotational kinematics while knowing that the initial *angular velocity* is zero and the *angular acceleration* is 4.4 rad/sec^2, we get:

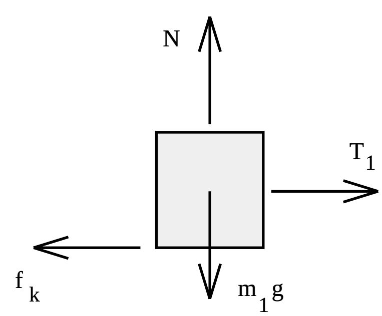
$$\begin{aligned}\Delta\theta &= \omega_1 t + (1/2)\alpha t^2\\ &= 0 \quad + .5(4.42\text{ rad/s}^2)(5\text{ s})^2\\ &= 55.25\text{ rad.} \qquad\qquad (\ldots\text{ close enough}).\end{aligned}$$

g.) Using kinematics:

$$\begin{aligned}\omega_5 &= \omega_1 + \qquad \alpha t\\ &= 0 \;+ (4.42\text{ rad/s}^2)(5\text{ s})\\ &= 22\text{ rad/sec} \qquad\qquad (\ldots\text{ close enough}).\end{aligned}$$

9.2) The thing to remember whenever you have a massive pulley is that the tension on either side of the pulley will be different.

a.) This is a Newton's Second Law problem. F.b.d.'s are shown to the right and on the next page, and N.S.L. is presented below (notice that you end up needing FIVE independent equations to solve this problem):



<u>for mass m_1:</u>

<u>$\sum F_y$:</u>

$$N - m_1 g = m_1 a_y = 0 \qquad (\text{as } a_y = 0)$$

$$\Rightarrow \quad N = m_1 g.$$

<u>$\sum F_x$:</u>

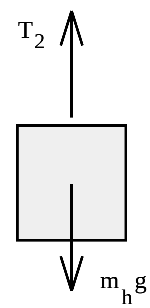
$$-\mu_k N + T_1 = m_1 a$$

$$\Rightarrow \quad T_1 = m_1 a + \mu_k m_1 g.$$

Note: This presents a problem. We don't want the acceleration a of the mass m_1, we want the angular acceleration α of the pulley. We need a relationship between those two quantities. Noticing that the string's acceleration a is that of m_1 and the string's acceleration is also the acceleration *of a point on the pulley's circumference*, we can use the relationship $a = R\alpha$ for the job. Doing so yields:

$$T_1 = m_1 \quad a \quad + \mu_k m_1 g$$
$$T_1 = m_1(R\alpha) + \mu_k m_1 g.$$

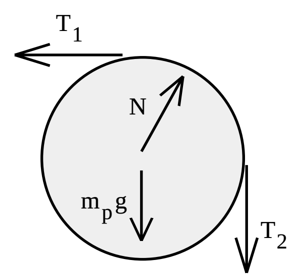
<u>for mass m_h:</u>



<u>$\sum F_y$:</u>

$$T_2 - m_h g = -m_h a$$
$$\Rightarrow \quad T_2 = -m_h \quad a \quad + m_h g$$
$$\Rightarrow \quad T_2 = -m_h(R\alpha) + m_h g.$$

<u>for the pulley:</u>



<u>$\sum \Gamma_{\text{pulley axis}}$:</u>

$$\Gamma_{T_1} - \Gamma_{T_2} = \quad - \quad I_p \quad \alpha$$
$$T_1 R - T_2 R = -[(1/2)M_p R^2]\alpha.$$

Note that the negative sign in front of the $I_p\alpha$ term denotes that the α term is a magnitude and the unembedded sign is designating a *clockwise angular acceleration*. Dividing out an R yields:

$$T_1 - T_2 = -[(1/2)M_p R]\alpha.$$

Substituting the expressions for T_1 and T_2 we get:

$$T_1 \qquad - \qquad T_2 \qquad = -[(1/2)M_p R]\alpha$$
$$[m_1(R\alpha) + \mu_k m_1 g] - [-m_h(R\alpha) + m_h g] = -[(1/2)M_p R]\alpha$$
$$\Rightarrow \quad \alpha = [-\mu_k m_1 g + m_h g] / [m_1 R + m_h R + .5 M_p R].$$

Put in the numbers and you should come out with 29.32 rad/sec^2.

b.) The acceleration of a point on the edge of the pulley (this is also the acceleration of the string which, in turn, is the acceleration of the hanging mass), is $a = r\alpha$, where r in this case is the radius R of the pulley and α is provided from *Part a* above. Doing the work yields:

$$\begin{aligned} a &= R\alpha \\ &= (.1875 \text{ m/rad})(29.32 \text{ rad/sec}^2) \\ &= 5.5 \text{ m/s}^2. \end{aligned}$$

c.) The total work done by all the *tension* forces in the system equals zero. Noting that, the *conservation of energy* yields:

$$\begin{aligned} \Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} &= \Sigma KE_2 + \Sigma U_2 \\ 0 + m_h gh + (-fh) &= [(1/2)m_1 v^2 + (1/2)m_h v^2 + (1/2)I_{cm}\omega^2] + 0. \end{aligned}$$

Noting that the velocity of a point on the pulley's circumference will equal v, and that $v = R\omega$, and the frictional force is $\mu_k N = \mu_k m_1 g$, we can write:

$$m_h gh = [(\mu_k m_1 gh) + (1/2)m_1(R\omega)^2 + (1/2)m_h(R\omega)^2 + (1/2)[I_{cm}\omega^2].$$

Rewriting this, eliminating the units (for space):

$$(1.2)(9.8)(1.5) = (.7)(.4)(9.8)(1.5) + .5(.4)(.1875)^2\omega^2 + .5(1.2)(.1875)^2\omega^2 + .5[1.4\text{x}10^{-3}]\omega^2$$

$$\Rightarrow \quad \omega = 21.66 \text{ rad/sec.}$$

d.) Noting that the string will have a velocity equal to that of both m_1 and m_h AND to a point on the circumference of the pulley, we can relate the angular velocity of the pulley and the string's velocity by $v = R\omega$. Remembering that the units for R in this usage are "meters/radian," we get:

$$\begin{aligned} v &= R\omega \\ &= (.1875 \text{ m/rad})(21.66 \text{ rad/sec}) \\ &= 4.06 \text{ m/s}. \end{aligned}$$

e.) This is a trick question. The acceleration is going to be the same no matter how far the hanging mass has fallen. The answer was derived in *Part b*.

f.) Angular momentum is defined rotationally as $L = I\omega$. With that we get:

$$L = I\omega$$
$$= [(1/2)m_pR^2]\,\omega$$
$$= .5(.08 \text{ kg})(.1875 \text{ m})^2(21.66 \text{ rad/sec})$$
$$= .03 \text{ kg}\cdot\text{m}^2/\text{s}.$$

9.3) An f.b.d. for the beam is shown below.

a.) This is a rigid body (i.e., equilibrium) problem. The easiest way to get the tension in the line is to sum the torques about the pin (that will eliminate the need to deal with *H* and *V* and will additionally give you an equation that has only one unknown--the *T* variable you are looking for).

Note:

--Call the distance between the pin and the cable's connection to the beam *d* = *(2/3)L*, where *L* is the beam's length.

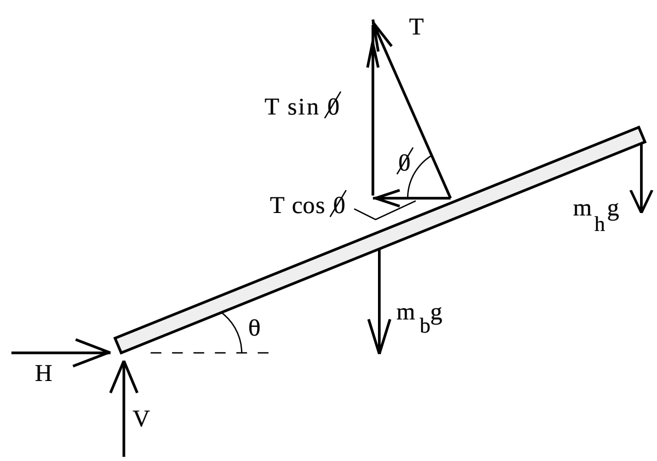
--The angle between *T* and *d* is 90^o which means the torque due to *T* about the pin will be $Td \sin 90^o = T(2/3)L$.

--The distance between the pin and the hanging mass's connection to the beam is *L*.

--The component of *L* perpendicular to the *line of the hanging mass* (i.e., *r-perpendicular*) will be $L \cos 30^o$ (this is like determining the shortest distance between the pin and the *line of the force*). That means the torque provided by the hanging mass will be $(m_hg)L\cos 30^o$. As this torque will try to make the beam rotate clockwise, it is a *negative* torque.

--The *beam's mass* can be assumed to be located at the beam's *center of mass* at *L/2*.

Putting all this together, N.S.L. yields:



$\underline{\sum \Gamma_{pin}}$:

$$\Gamma_T \quad + \quad \Gamma_{m_h} \quad + \quad \Gamma_{m_b} \quad = 0 \quad (\text{as } \alpha = 0)$$

$$T[(2/3)L] - (m_hg)L\cos 30^o - (m_bg)(L/2)\cos 30^o = 0.$$

Canceling the *L's* and solving for *T*, we get:

$$T = [(m_h g)\cos 30^o + (m_b g)(.5)\cos 30^o]/(2/3)$$
$$= [(3\text{ kg})(9.8\text{ m/s}^2)(.87) + (7\text{ kg})(9.8\text{ m/s}^2)(.5)(.87)]/[.67]$$
$$= 82.7\text{ nts.}$$

Knowing *T*, we can sum the forces in the *x* and *y directions* to determine *H* and *V*. Doing so yields:

$\underline{\sum F_x}$:

$$-T\cos 60^o + H = ma_x$$
$$= 0 \quad \text{as } a_x = 0.$$
$$\Rightarrow \quad H = T\cos 60^o$$
$$= (82.7\text{ nt})(.5)$$
$$= 41.35\text{ nts.}$$

$\underline{\sum F_y}$:

$$T\sin 60^o + V - m_h g - m_b g = ma_y$$
$$= 0 \quad (\text{as } a_y = 0)$$
$$\Rightarrow \quad V = -T\sin 60^o + m_h g + m_b g$$
$$= -(82.7\text{ nt})(.87) + (3\text{ kg})(9.8\text{ m/s}^2) + (7\text{ kg})(9.8\text{ m/s}^2)$$
$$= 26.05\text{ nts.}$$

Note: If there had been a negative sign in front of the 26.05 newtons, it would have meant that we had assumed the wrong direction for the force *V*. The *magnitude* would nevertheless have been correct.

b-i.) The *moment of inertia* about an axis other than one through the *center of mass* but parallel to an axis of known *moment of inertia* that is through the *center of mass* is determined using the *Parallel Axis Theorem*. The *moment of inertia* about the beam's pin is:

$$I_p = I_{cm} + mh^2$$
$$= (1/12)m_b L^2 + m_b (L/2)^2$$
$$= (1/3)m_b L^2$$
$$= (1/3)(7\text{ kg})(1.7\text{ m})^2$$
$$= 6.74\text{ kg}\cdot\text{m}^2.$$

b-ii.) The *moment of inertia* of a point mass (i.e., the hanging mass) *r* units from a reference axis (note that $r = L$ in this problem) is:

$$\begin{aligned} I_{hm} &= mr^2 \\ &= m_h L^2 \\ &= (3\text{ kg})(1.7\text{ m})^2 \\ &= 8.67\text{ kg}\cdot\text{m}^2. \end{aligned}$$

b-iii.) The total *moment of inertia* about the pin is the beam's *moment of inertia* about the pin added to the hanging mass's *moment of inertia* about the pin, or:

$$\begin{aligned} I_{tot,pin} &= \quad I_p \quad + \quad I_{hm} \\ &= (6.74\text{ kg}\cdot\text{m}^2) + (8.67\text{ kg}\cdot\text{m}^2) \\ &= 15.4\text{ kg}\cdot\text{m}^2. \end{aligned}$$

c.) Angular acceleration--think N.S.L. Summing up the torques *about the pin* (the beam is executing a pure rotation about the pin--no reason to sum torques about any other point) and putting that equal to the *moment of inertia* ABOUT THE PIN times the *angular acceleration* about the pin yields:

$\underline{\sum \Gamma_{pin}:}$

$$\Gamma_T \quad + \quad \Gamma_{m_{hm}} \quad + \quad \Gamma_{m_b} \quad = I_{tot,pin}\alpha$$

$$T[(2/3)L] - (m_h g)L\cos 30^o - (m_b g)(L/2)\cos 30^o = I_{tot,pin}\alpha.$$

The line has been cut (which means $T = 0$) and we know the total *moment of inertia* about the pin from above. Using this we get:

$$\Gamma_T \quad + \quad \Gamma_{m_h} \quad + \quad \Gamma_{m_b} \quad = \quad I_{tot,pin}\alpha$$

$$0 \quad - (m_h g)L\cos 30^o - (m_b g)(L/2)\cos 30^o = -(15.4\text{ kg}\cdot\text{m}^2)\alpha.$$

Solving for α we get:

$$\begin{aligned} \alpha &= [(m_h g)L\cos 30^o + (m_b g)(L/2)\cos 30^o)]/(15.4\text{ kg}\cdot\text{m}^2) \\ &= [(3\text{ kg})(9.8\text{ m/s}^2)(1.7\text{ m})(.87) + (7\text{ kg})(9.8\text{ m/s}^2)(.85\text{ m})(.87)]/(15.4\text{ kg}\cdot\text{m}^2) \\ &= 6.11\text{ rad/sec}^2. \end{aligned}$$

d.) Knowing the *angular acceleration* of the beam just after it lets loose allows us to determine the *translational acceleration* of any point on the beam using $a = r\alpha$, where *r* is the distance between the pin and the point-in-question. In this case, that distance is *L/2*. Using this information, we get:

$$\begin{aligned} a &= r\alpha \\ &= [(1.7 \text{ m})/2]\,(6.11 \text{ rad/sec}^2) \\ &= 5.19 \text{ m/s}^2. \end{aligned}$$

e.) This is a *conservation of energy* problem. The beam has *potential energy* wrapped up in the fact that its *center of mass* will have fallen a vertical distance equal to *(L/2) sin 30°* during the freefall. That means that its initial *potential energy* will be $m_b g(L/2)\sin 30^o$ while its final U will be zero. The hanging mass also falls a vertical distance equal to *L sin 30°*.

The *potential energy* changes of both bodies (i.e., the beam and hanging mass) must be taken into account if we are going to use *conservation of energy*--the approach of choice whenever we are looking for a velocity-type variable.

Assuming we approach the beam as though it were executing a pure rotation about its pin, we can relate the change in the potential energy of the system before the snip to the *rotational kinetic energy* of the beam about the pin plus the *translational kinetic energy* of the hanging mass just as the beam becomes horizontal. Using that approach:

$$\begin{array}{lllll} \sum KE_1 + & \sum U_1 & + \sum W_{ext} = & \sum KE_2 & + \sum U_2 \\ 0 \;+ & [m_b g((L/2)\sin 30^o) + m_h g(L \sin 30^o)] & + 0 \;= & [(1/2)m_h v^2 + (1/2)I_p\omega^2] & + \; 0. \end{array}$$

Noting that the velocity of a point on the beam will equal $v = R\omega$, we can write the velocity of the hanging mass as $v_h = (L)\omega$:

$$m_b g((L/2)\sin 30^o + m_h g(L \sin 30^o) = (1/2)m_h \; v_h^{\,2} + (1/2) \quad I_p \quad \omega^2$$

$$m_b g((L/2)\sin 30^o + m_h g(L \sin 30^o) = (1/2)m_h[L\omega]^2 + (1/2)[(1/3)m_b L^2]\,\omega^2.$$

Putting in the numbers while eliminating the units (for space), we get:

$$(7)(9.8)(.85)(.5) + (3)(9.8)(1.7)(.5) = .5(3)(1.7^2)\,\omega^2 + .5(.33)(7)(1.7)^2\,\omega^2$$

$$\Rightarrow \quad \omega = 2.62 \text{ rad/sec.}$$

f.) The *translational velocity* of the *center of mass* will be:

$$v = (L/2)\ \omega$$
$$= [(1.7\text{ m})/2](2.62\text{ rad/sec})$$
$$= 2.23\text{ m/s}.$$

g.) *Angular momentum* is defined rotationally as $L = I\omega$, where I is the TOTAL *moment of inertia* for the beam and hanging mass about the pin, and the fact that the length of the beam has been defined as L turns out to be a really, really bad choice of variables (it's the same as *angular momentum*). Remembering that $I_{tot,pin} = 15.4\ kg\cdot m^2/s$, we get:

$$L = I\omega$$
$$= [15.4](2.62\text{ rad/sec})$$
$$= 40.35\text{ kg}\cdot\text{m}^2\text{/s}.$$

h.) *Angular momentum* is conserved *if and only if* all the torques acting on the system are internal (i.e., are the consequence of the interaction of the various parts of the system). Gravity is an external force which means that any torque it produces on the beam will be an *external* torque. In short, *angular momentum* should not be conserved.

(This should be obvious given the fact that the beam is accelerating angularly).

9.4) The merry-go-round's mass is $m_m = 225\ kg$ while each child's mass is $m_c = 35\ kg$. The radius of the merry-go-round is $R = 2.5$ *meters* and its *angular velocity* is $\omega_1 = .8$ *radians/second* when the kids climb on.

a.) An *angular velocity* of $\omega_1 = .8\ rad/sec$ produces a *translational velocity* of $r\omega = (2.5\ m)(.8\ rad/sec) = 2\ m/s$ at the edge of the merry-go-round (i.e., where the children are when they first jump on). Also, the *moment of inertia* of one child when he or she first jumps onto the merry-go-round is $I_{c,1} = m_cR^2 = (35\ kg)(2.5\ m)^2 = 218.75 kg\cdot m^2$ while the *moment of inertia* of the merry-go-round itself is assumed to be that of a disk and is equal to $I_m = (1/2)m_mR^2 = .5(225\ kg)(2.5\ m)^2 = 703\ kg\cdot m^2$. Using *conservation of energy,* we can write:

$$\Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} = \Sigma KE_2 + \Sigma U_2$$
$$0 + 0 + 3[(RF)\Delta\theta] = [3(1/2)I_c\omega_2^{\ 2} + (1/2)I_m\omega_2^{\ 2}] + 0.$$

Note: $\Gamma_{one\ kid\ on\ mgr} = RF\sin 90^o$. The counterpart to $\mathbf{F}\cdot\mathbf{d}$ for rotational motion is $\mathbf{\Gamma}\cdot\Delta\boldsymbol{\theta}$, so the three kids do $3[(RF)\Delta\theta]$ worth of extra work on the m.g.r.

Solving, we get:

$$3(2.5\text{ m})(15\text{ nt})\Delta\theta = [3(.5)(218.75\text{ kg}\cdot\text{m}^2)(.8\text{ rad/s})^2 + .5(703\text{ kg}\cdot\text{m}^2)(.8\text{ r/s})^2]$$
$$\Rightarrow \quad \Delta\theta = 3.87\text{ radians.}$$

Note that the relationship between angular displacement and linear displacement is:

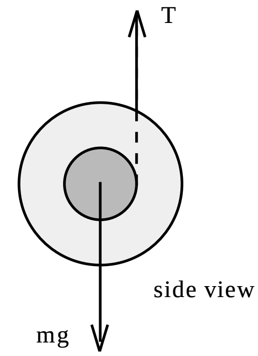
$$\Delta s = R\Delta\theta$$
$$= (3.87\text{ rad})(2.5\text{ m/rad})$$
$$= 9.68\text{ meters.}$$

b.) Once the kids have climbed aboard, the torques acting on the system are all internal. That is, they are due to the interaction of the system's parts (the kids act on the merry-go-round while the merry-go-round acts on the kids). That means we can use the *conservation of <u>angular momentum</u>:*

$$L_{1,tot} = L_{2,tot}$$
$$L_{m,1} + L_{c,1} = L_{m,2} + L_{c,2}$$
$$I_m\omega_1 + (I_{c,1}\omega_1) = I_m\omega_2 + [I_{c,2}\omega_2]$$
$$(703\text{ kg}\cdot\text{m}^2)(.8\text{ r/s}) + (656\text{ kg}\cdot\text{m}^2)(.8\text{ r/s}) = (703\text{ kg}\cdot\text{m}^2)(\omega_2) + [3(35\text{ kg})(1\text{ m})^2\ \omega_2]$$
$$\Rightarrow \quad \omega_2 = 1.35\text{ rad/sec and}$$
$$\Rightarrow \quad v_2 = r\omega$$
$$= (1\text{ m/rad})(1.346\text{ rad/sec})$$
$$= 1.35\text{ m/s.}$$

c.) As there are no external torques, the *angular momentum* should be conserved.

Also, the forces acting on the system are *internal* so the *total momentum* of the system will be conserved as the children move toward the merry-go-round's center. It should be noted, though, that this isn't a very useful fact as far as problem-solving goes. The addition of the momentum components of the children's motion at the beginning *and at any time thereafter* will be zero.



d.) The forces being exerted on the system are not conservative (the kids burn chemical energy as they move on the merry-go-round), so energy is not conserved.

e.) We have already deduced that energy should not be conserved. Let's see if we were right by calculating the energy in the system just after the kids jumped onto the merry-go-round and the energy once they reached the *r = 1 meter* point:

$$KE_1 = KE_{c,1} + KE_{m,1}$$
$$= (1/2)m_{c,1}v_1^{\,2} + (1/2)I_m\omega_1^{\,2}$$
$$= .5[3(35\text{ kg})](2\text{ m/s})^2 + .5(703\text{ kg}\cdot\text{m}^2)(.8\text{ r/s})^2$$
$$= 435\text{ joules.}$$

$$KE_2 = KE_{c,2} + KE_{m,2}$$
$$= (1/2)m_{c,2}v_2^{\,2} + (1/2)I_m\omega_2^{\,2}$$
$$= .5[3(35\text{ kg})](1.35\text{ m/s})^2 + .5(703\text{ kg}\cdot\text{m}^2)(1.35\text{ r/s})^2$$
$$= 736.3\text{ joules.}$$

Energy is obviously not conserved.

9.5) Among other reasons, this problem was designed to allow you to see that either the *pure rotation* or the *rotation about the center of mass plus translation of the center of mass* approaches will work when dealing with either a Newton's Second Law or *conservation of energy* problem. An f.b.d. for the problem is shown to the right.

a.) Analyzing the wheel problem from a *rotation about the center of mass plus translation of the center of mass* approach:

This is a N.S.L. problem. We will begin by summing the torques about the center of mass:

$$\underline{\sum \Gamma_{cm}:}$$
$$Tr_a = I_{cm}\alpha \qquad \text{(Equation A).}$$

We have two unknowns here; we need another equation. Consider the sum of the forces in the vertical:

$\underline{\sum F_v:}$

$$T - mg = -ma$$
$$= -m(r_a \alpha)$$

$$\Rightarrow \quad T = mg - mr_a \alpha \qquad \text{(Equation B).}$$

Putting *Equations A* and *B* together:

$$Tr_a = I_{cm} \alpha$$
$$\Rightarrow \quad (mg - mr_a \alpha)r_a = I_{cm} \alpha$$
$$\Rightarrow \quad \alpha = (mgr_a)/(I_{cm} + mr_a{}^2)$$
$$= [(.6 \text{ kg})(9.8 \text{ m/s}^2)(.015 \text{ m})]/[(1.2\text{x}10^{-4} \text{ kg}\cdot\text{m}^2) + (.6 \text{ kg})(.015)^2]$$
$$= 346 \text{ rad/sec}^2.$$

b.) We have to use the Parallel Axis Theorem to determine the *moment of inertia* about the new axis. Doing so yields:

$$I_a = \quad I_{cm} \quad + \quad m \quad h^2$$
$$= (1.2\text{x}10^{-4} \text{ kg}\cdot\text{m}^2) + (.6 \text{ kg})(.015 \text{ m})^2$$
$$= 2.55\text{x}10^{-4} \text{ kg}\cdot\text{m}^2.$$

c.) Analyzing the wheel problem from a *pure rotation* approach:

This is a N.S.L. problem. We will begin by summing the torques about the axis .015 meters from the *center of mass* (call this *Point P*).

$\underline{\sum \Gamma_a:}$

$$(mg)r_a = I_a \alpha.$$

We determined I_a in *Part b*, so all we have to do is solve for α:

$$\alpha = (mgr_a)/I_a$$
$$= [(.6 \text{ kg})(9.8 \text{ m/s}^2)(.015 \text{ m})]/(2.55\text{x}10^{-4} \text{ kg}\cdot\text{m}^2)$$
$$= 346 \text{ rad/sec}^2.$$

Great jumping Huzzahs! *Parts a* and *c* match. Both approaches work.

d.) This is a *conservation of energy* problem. Major note: *Work* requires that a force act *through a distance*. Tension acts at the one point on the disk that *isn't moving*--it acts at the instantaneously stationary point about which the disk rotates. As such, tension acts through *no distance* and does *no work*.

--from the *pure rotation* approach:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$0 + mgd + 0 = (1/2)I_a\omega^2 + 0$$
$$0 + (.6\text{ kg})(9.8\text{ m/s}^2)(.18\text{ m}) + 0 = .5(2.55\text{x}10^{-4}\text{ kg}\cdot\text{m}^2)\omega^2 + 0$$
$$\Rightarrow \quad \omega = 91.1\text{ rad/sec.}$$

--from the *rotation about the center of mass plus translation of the center of mass* approach, noting that $v_{cm} = r_a\omega$:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$0 + mgd + 0 = [(1/2)I_{cm}\omega^2 + (1/2)mv_{cm}^2] + 0$$
$$0 + (.6\text{ kg})(9.8\text{ m/s}^2)(.18\text{ m}) + 0 = [.5(1.2\text{x}10^{-4}\text{ kg}\cdot\text{m}^2)\omega^2 + .5(.6\text{ kg})[(.015\text{ m})\omega]^2] + 0$$
$$\Rightarrow \quad \omega = 91.1\text{ rad/sec.}$$

Again, we get the same answer no matter which approach we use.

e.) This is trivial, given the fact that you know the *angular velocity* calculated in *Part d*.

$$\begin{aligned} v_{cm} &= r_a\omega \\ &= (.015\text{ m/rad})(91.1\text{ rad/sec}) \\ &= 1.37\text{ m/s.} \end{aligned}$$

9.6)

a.) As the *force per unit length* function is $\xi = kx$, it makes sense that k times ξ must have the units *newtons per meter*. This will be the case if k has the units nt/m^2.

b.) This is a N.S.L. problem. The moment of inertia of a rod about one end is $(1/3)mL^2$, where m is the rod's mass and L is its length. Since a beam is simply a squared off rod, we will take the moment of inertia of the beam about its pin to be *(1/3)*mL^2. There are two torques acting about the pin. To do the problem we need to determine those torques.

--The magnitude of the torque about the pin due to the beam's weight acting at the beam's center of mass (i.e., at *L/2*) is:

$$\Gamma_{mg} = (mg)(L/2)\sin 90^o.$$

--The magnitude of the torque about the pin due to the differential bits of force acting downward on the beam along its length is more difficult to determine. To do so, follow along while considering the figure below.

--The magnitude of the differential force *dF* acting over a differential length *dx* applied a distance *x* meters from the pin will be:

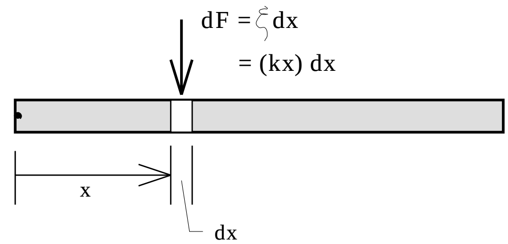
$$\begin{aligned} dF &= \xi\, dx \\ &= (kx)\, dx. \end{aligned}$$

--The magnitude of the differential torque $d\Gamma$ about the pin applied by the differential force *dF* is:

$$\begin{aligned} d\Gamma &= |\mathbf{r}|\quad|\mathbf{F}|\quad \sin 90^o \\ &= x\ (kx\, dx)\ \ (1) \\ &= kx^2\, dx. \end{aligned}$$

--The net torque about the pin will be the sum of the differential torques (i.e., those applied due to the presence of the force distributed along the beam's length) and the torque due to gravity. Remembering that the moment of inertia about the beam's pin is I_{pin} = *(1/3)*mL^2:

differential force on beam



$\underline{\Sigma\Gamma_{pin}:}$

$$-(mg)(L/2) - \int_{x=0}^{L}(kx^2)dx = -I_{pin}\alpha$$

$$\Rightarrow \alpha = \frac{(mg)(L/2) + k\left(\frac{L^3}{3}\right)}{\left(\frac{1}{3}\right)mL^2}$$

$$= \frac{3g}{2L} + \frac{kL}{m}.$$

9.7)

a.) This is a collision problem. *Energy* is not conserved through the collision because non-conservative forces act during the collision and because the collision was not close enough to being lossless to be approximated as *elastic*. *Momentum* is not conserved because the pin applies an external force to the rod. *Angular momentum* <u>is</u> conserved as there are no external torques (the external force at the pin applies no torque to the system because the pin force is applied at the axis of rotation).

As the rod is massless, we can treat all of the masses in the system as point masses. We know that there are two ways to calculate angular momentum (i.e., by using the magnitude of ***rx(mv)*** or by using $I\omega$), and we know that the *moment of inertia* of a point mass is mr^2 (this means the total moment of inertia of the system is $m_1(d/2)^2 + 2m_2(d/2)^2 = (m_1 + 2m_2)(d/2)^2$). With all of this, we can write:

$$\Sigma L_o = \Sigma L_f$$

$$L_{wad,o} + 2L_{mass,o} = I_{tot}\ \omega_1$$

$$m_1 v_o (d/2) \sin 90^o + 0 = [(m_1 + 2m_2)(d/2)^2]\ \omega_1.$$

Plugging in the numbers, we get:

$$(.9\text{ kg})(2.8\text{ m/s})(1.2/2\text{ m}) \sin 90^o = [(.9\text{ kg} + 2(2\text{ kg}))(1.2/2\text{ m})^2]\,\omega_1$$

$$\Rightarrow \quad \omega_1 = .857\text{ rad/sec.}$$

$$\Rightarrow \quad v_{at\ ends} = (d/2)\,\omega_1$$

$$= (1.2/2\text{ m})(.857\text{ rad/sec})$$

$$= .514\text{ m/s.}$$

b.) As there is essentially no *potential energy* change through the collision, the *total energy change* will be wrapped up in the *kinetic energy* difference. The initial *kinetic energy* is all translational, being associated with the wad. The final *kinetic energy* can be treated as purely rotational. Doing so yields an energy difference of:

$$\begin{aligned}\Delta KE &= KE_{after} - KE_{bef} \\ &= (1/2)\, I_{tot}\, \omega_1^{\,2} - (1/2)(m_1)v_o^{\,2} \\ &= .5\,[(m_1 + 2m_2)(d/2)^2]\, \omega_1^{\,2} - .5(m_1)v_o^{\,2}.\end{aligned}$$

Putting in the numbers we get:

$$\begin{aligned}\Delta KE &= .5[(.9 \text{ kg} + 2(2 \text{ kg}))(1.2/2 \text{ m})^2](.857 \text{ rad/sec})^2 - .5(.9 \text{ kg})(2.8 \text{ m/s})^2 \\ &= -2.884 \text{ joules} \quad (\text{energy is lost}).\end{aligned}$$

c.) The net angular displacement of the system can be determined using *conservation of energy* for the motion *after the collision*. In fact, the symmetry of the problem makes the calculation particularly easy as the *potential energy* picked up by the *right mass* in its transit is equal to the potential energy lost by the *left mass*. As such, the only *potential energy* gain will be associated with the position-change of the wad. That means:

$$\begin{aligned}\Sigma KE_1 + \Sigma U_1 + \Sigma W_{ext} &= \Sigma KE_2 + \Sigma U_2 \\ (1/2)I_{tot}\,\omega_1^{\,2} + 0 + 0 &= 0 + m_1 gh.\end{aligned}$$

where h is the vertical distance above the wad's initial position. Solving for h yields:

$$\begin{aligned}h &= (1/2)I_{tot}\,\omega_1^{\,2}/(m_1 g) \\ &= (1/2)[(.9 \text{ kg} + 2(2 \text{ kg}))(1.2/2 \text{ m})^2](.857 \text{ rad/sec})^2 / [(.9 \text{ kg})(9.8 \text{ m/s}^2] \\ &= .073 \text{ meters.}\end{aligned}$$

With h, we can form a right triangle whose hypotenuse is $d/2 = .6$ *meters* and whose *opposite side* is $h = .073$ *meters* (see the triangle formed in Figure IV on the previous page). Taking the *inverse sine* yields an angle of .122 radians. That means the rod's net angular displacement

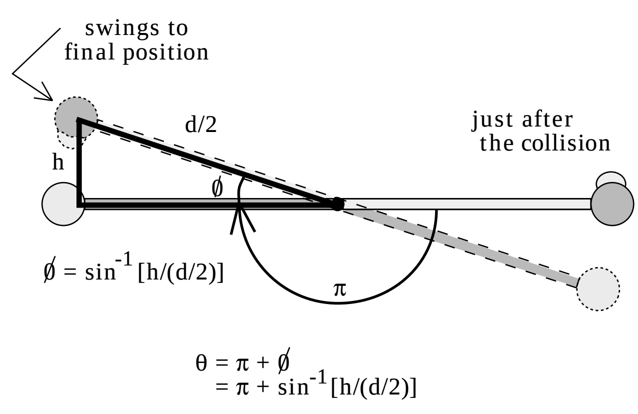


FIGURE IV

will be π radians plus .122 radians, or 3.26 radians.

Note: As we know the energy lost from *Part b*, we could have used the *modified conservation of energy* equation *through the collision*. Doing so would have yielded:

$$\begin{aligned} \Sigma KE_o \quad + \quad \Sigma U_o + \Sigma W_{ext} &= \Sigma KE_1 + \Sigma U_1 \\ (1/2)m_1 v_o^2 \quad + \quad 0 \quad + (-2.884 \text{ j}) &= \quad 0 \quad + m_1 gh \\ \Rightarrow \quad (1/2)(.9 \text{ kg})(2.8 \text{ m/s})^2 + \quad (-2.884 \text{ j}) &= (.9 \text{ kg})(9.8 \text{ m/s}^2)h \\ \Rightarrow \quad h &= .073 \text{ meters.} \end{aligned}$$

9.8) This obviously has within it a collision problem. Through the collision, *energy* is not conserved because the forces involved in the collision are undoubtedly non-conservative and because the problem's author has not deemed it necessary to imbue the collision with the magical label of *elastic*. *Momentum* is not conserved due to the fact that an external force is provided by the pin. *Angular momentum* is conserved because the external force at the pin produces no torque on the system.

If we can determine the amount of energy there is in the system just after the collision, we can use conservation of energy to determine the rise of the stick's center of mass after the collision and, from that, the final angle of the stick. As the *conservation of angular momentum* will allow us to determine the angular velocity of the system just after the collision and, with that, the energy in the system just after the collision, we will begin there.

a.) Assume the before-collision velocity of the block (i.e., after free falling to the bottom of the incline) is v_1 and the after-collision angular velocity of the stick is ω_2. Also, assume the block stays stationary just after the collision. With all that, the angular momentum about the pin is:

$$\begin{aligned} L_{before} &= L_{after} \\ L_{block,bef} + L_{rod,bef} &= L_{rod,aft} + L_{block,aft} \\ mv_1 d \sin 90^o + \quad 0 &= I_{p,rod}\ \omega_2 + \quad 0 \\ &= [(1/3)(5m)d^2]\,\omega_2 \\ \Rightarrow \quad \omega_2 &= (3/5)v_1/d. \end{aligned}$$

The problem here? We don't know what v_1 is. To determine that quantity, use *conservation of energy* during the block's slide down the frictionless incline. Doing so yields:

$$\sum KE_o + \sum U_o + \sum W_{ext} = \sum KE_1 + \sum U_1$$
$$0 + mg(.4d) + 0 = (1/2)mv_1^{\,2} + 0$$
$$\Rightarrow \quad v_1 = [2g(.4d)]^{1/2}$$
$$= 2.8(d)^{1/2}.$$

With v_1, the angular velocity of the stick just after the collision becomes:

$$\omega_2 = (3/5)v_1/d$$
$$= (3/5)[2.8(d)^{1/2}/d]$$
$$= 1.68/(d)^{1/2}.$$

We are now in a position to use *conservation of energy* from the time just after the collision to the time when the stick gets to the top of its motion (see Figure V). Defining the position of both the center of mass of the stick and the position of the block *just after the collision* to be the *potential energy equals zero* level for each object, we can write:

$$\sum KE_2 + \sum U_2 + \sum W_{ext} = \sum KE_3 + \sum U_3$$
$$[KE_{2,bl} + KE_{2,st}] + [U_{2,bl} + U_{2,st}] + \sum W_{ext} = [KE_{3,bl} + KE_{3,st}] + [U_{3,bl} + U_{3,st}]$$
$$[\ 0 + (1/2)I_p\omega_2^{\,2}\ ] + [\ 0 + 0\ ] + 0 = [\ 0 + 0\ ] + [\ 0 + (5m)gh\ ].$$

Solving:

$$\Rightarrow \quad .5 \quad I_{p,st} \quad \omega_2^{\,2} = (5m)\,g \quad h$$
$$.5[(1/3)(5m)d^2]\ [1.68/(d)^{1/2}]^2 = (5m)\,g\,[(d/2)(1 - \cos\theta)]$$
$$\Rightarrow \quad \theta = .44 \text{ radians or } 25.3^\circ.$$

b.) Assume the block adheres to the stick after the collision. Assume also that the before-collision velocity of the block (i.e., after free falling to the bottom of the incline) is v_1, the block's *moment of inertia* about the pin is md^2, and the after-collision angular velocity of both the block and stick is ω_2. With all this, we can write:

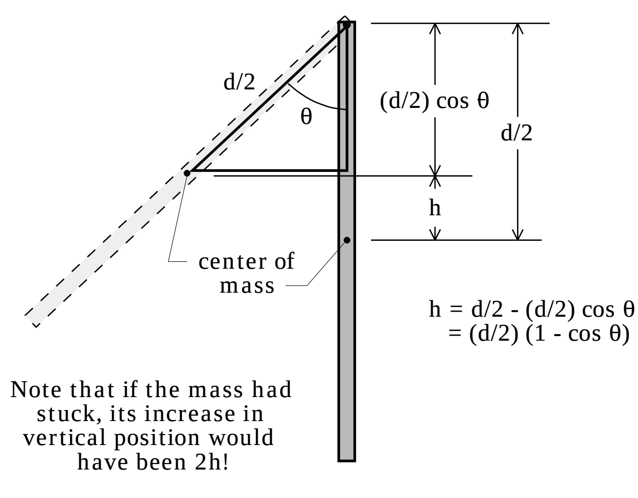


FIGURE V

$$L_{before} = L_{after}$$
$$L_{block,bef} + L_{rod,bef} = L_{rod,aft} + L_{block,aft}$$
$$mv_1 d \sin 90^o + 0 = I_{pin,rod}\,\omega_2 + I_{p,bl}\,\omega_2$$
$$\Rightarrow \quad mv_1 d = [(1/3)(5m)d^2 + md^2]\,\omega_2.$$

Canceling *d's* and *m's* and simplifying, we get:

$$\omega_2 = .375 v_1/d.$$

What is v_1? As derived in *Part a*:

$$v_1 = 2.8d^{1/2}.$$

With v_1, the angular velocity of the stick just after the collision becomes:

$$\begin{aligned}\omega_2 &= .375 \quad v_1 \quad /d \\ &= .375[2.8(d)^{1/2}]/d \\ &= 1.05/d^{1/2}.\end{aligned}$$

We are now in a position to use *conservation of energy* from the time just after the collision to the time when the stick gets to the top of its motion (Figure V still captures the spirit of this calculation). Defining the position of the center of mass of the meter stick and the position of the block *just after the collision* to be the *potential energy equals zero* levels for each object, remembering that the moment of inertia of the block is md^2, and noting that if the stick's center of mass rises a distance h, the block rises a distance $2h$ (see Figure V), we can write:

$$\sum KE_2 + \sum U_2 + \sum W_{ext} = \sum KE_3 + \sum U_3$$
$$[KE_{2,bl} + KE_{2,st}] + 0 + 0 = [KE_{3,bl} + KE_{3,st}] + [U_{3,bl} + U_{3,st}]$$
$$[.5(md^2)\omega_2^2 + .5I_p\omega_2^2] + 0 + 0 = [\ 0 + 0\ ] + [mg(2h) + (5m)gh].$$

Noting that the stick's *moment of inertia* about the pin is $I_p = (1/3)(5m)d^2 = (5/3)md^2$, we can simplify to get:

$$.5md^2\omega_2^2 + .5[(5/3)md^2]\omega_2^2 = mg[2(d/2)(1 - \cos\theta)] + (5m)g[(d/2)(1 - \cos\theta)]$$
$$\Rightarrow \quad [.5md^2 + (5/6)md^2]\,\omega_2^2 = [2mg + (5m)g]\ [(d/2)(1 - \cos\ \theta)]$$
$$\Rightarrow \quad 1.33md^2\omega_2^2 = (7/2)mgd(1 - \cos\ \theta)].$$

Using $\omega_2 = 1.05/d^{1/2}$, we find:

$$1.33md^2[1.05/d^{1/2}]^2 = (7/2)mgd - (7/2)mgd\cos\theta.$$

Canceling *m's* and the *d's*, we get:

$$1.33[1.05^2] = (7/2)g - (7/2)g\cos\theta$$
$$\Rightarrow \quad \theta = .293 \text{ radians or } 16.8^o.$$

9.9) This is a slippage problem in which nothing is conserved. Approach such problems with N.S.L. Treat the frictional force involved as an average, and define the time it takes f_{avg} to bring the two bodies into a pure roll as Δt.

The f.b.d. for the situation is shown in Figure VI to the right.

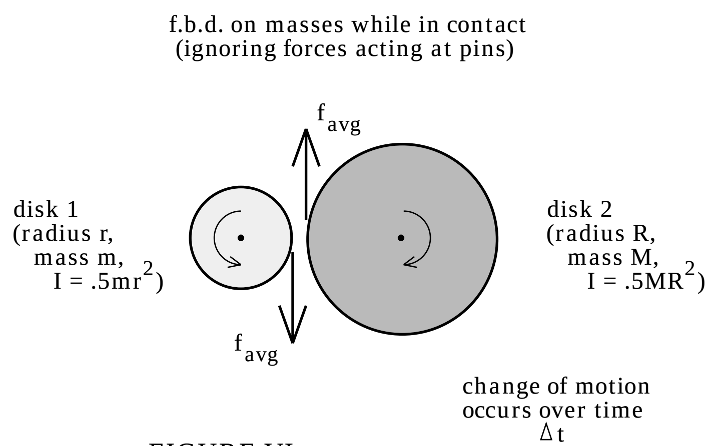


FIGURE VI

Summing the torques acting on the small mass, we get:

$$\underline{\sum \Gamma_{cm,1}}:$$
$$-f_{avg}r = I_{1,m}\left(\frac{\omega_3 - \omega_0}{\Delta t}\right)$$
$$\Rightarrow \quad -f_{avg}\Delta t = \left(.5mr^2\right)\left(\frac{\omega_3 - \omega_0}{r}\right)$$
$$\Rightarrow \quad -f_{avg}\Delta t = (.5mr)(\omega_3 - \omega_0) \quad \text{(Equ.A)}.$$

In doing a similar summation on the large disk, we will have to unembed the required negative sign inherent within the angular velocity term (we didn't have to do that for *m* because its angular velocities were both positive and the required negative sign was provided by the fact that $\omega_o > \omega_3$). That calculation yields:

$\underline{\sum \Gamma_{cm,2}:}$

$$-f_{avg}R = -I_{1,M}\left(\frac{\omega_4 - 0}{\Delta t}\right)$$

$$\Rightarrow \quad f_{avg}\Delta t = \left(.5MR^2\right)\left(\frac{\omega_4}{R}\right)$$

$$\Rightarrow \quad f_{avg}\Delta t = \left(.5MR\right)\omega_4 \qquad \text{(Equ.B).}$$

Adding *Equations A* and *B* yields:

$$0 = .5mr\,\omega_3 - .5mr\,\omega_o + .5MR\,\omega_4.$$

We need a way to relate ω_3 to ω_4. This can be done by noting that when the disks are finally rolling without slippage, the translational velocity of each disk *at the point of contact* must be the same (otherwise, there would be slippage). That means:

$$v_{3,edge} = r\omega_3 = V_{4,edge} = R\,\omega_4$$
$$\Rightarrow \quad \omega_4 = (r/R)\,\omega_3.$$

Using this, we can write:

$$0 = .5mr\,\omega_3 - .5mr\,\omega_o + .5MR\,\omega_4$$
$$\Rightarrow \quad 0 = .5mr\,\omega_3 - .5mr\,\omega_o + .5MR(r/R)\,\omega_3$$
$$\Rightarrow \quad \omega_3 = \frac{m}{(m+M)}w_o.$$